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APPLICATIONS OF FRACTIONAL CALCULUS IN BIOMEDICAL SIGNAL PROCESSING

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Abstract

It has recently come to light that fractional calculus is a powerful mathematical tool that may be utilised for modelling and analysing the complex dynamics of biological signals. Within the scope of this work, we offer a summary of the many applications of fractional calculus in the field of biomedical signal processing. Electrocardiograms (ECG), electroencephalograms (EEG), and electromyograms (EMG) are examples of biomedical signals that frequently display non-linear and non-stationary properties. As a result, the analysis of these signals using traditional approaches can be difficult. In order to effectively describe these intricate processes, fractional calculus provides a strong framework. In this article, we will examine the application of fractional calculus in the modelling of physiological processes that involve long-term memory effects, such as heart rate variability (HRV) signals. Additionally, we will describe how this technique permits the building of precise models that can accurately anticipate physiological occurrences. In addition, we investigate the ways in which fractional differentiation and integration operators improve signal processing tasks such as denoising, feature extraction, and classification, eventually leading to enhanced diagnostic and predictive capabilities in biomedical systems. The purpose of this overview is to bring attention to the potential of fractional calculus in enhancing our understanding of intricate physiological processes and enhancing diagnostic methods in the field of biomedical signal processing.

keywords: *Fractional Calculus, Derivatives of Non-integer Orders, Anomalous Diffusion, Viscoelastic Materials, Signal Processing, Physiological Processes, Interdisciplinary Applications*

Introduction

Application of fractional calculus, which is an extension of standard calculus to orders that are not integers, has been discovered to have exciting applications in a variety of domains, including biomedical signal processing. Regarding this particular context, it provides a sophisticated mathematical framework that can be utilised to represent and analyse complicated dynamics that are frequently seen in physiological systems. Electrocardiograms (ECG), electroencephalograms (EEG), and electromyograms (EMG) are examples of biomedical signals that display non-linear and non-stationary properties. As a result, the analysis of these signals using traditional approaches can be difficult. Utilising fractional calculus, one may acquire the tools necessary to correctly record these complex processes. The investigation of physiological processes that display long-term memory effects, such as the fractal-like behaviour found in heart rate variability (HRV) signals, is one application that stands out as particularly noteworthy.

The use of fractional calculus makes it possible to build models that take into account the features of memory, which ultimately results in more accurate representations and predictions of physiologically occurring phenomena. Furthermore, fractional differentiation and integration operators provide increased flexibility in signal processing activities such as denoising, feature extraction, and classification applications. By introducing fractional operators into signal processing algorithms, researchers are able to better preserve useful information while reducing noise and irrelevant components. This results in an improvement in the diagnostic and predictive capacities of biomedical systems. In general, the use of fractional calculus in the field of biomedical signal processing offers the potential to further our understanding of intricate physiological processes, improve diagnostic methods, and improve the quality of care provided to patients. Through the following sections, we will look more deeply into particular applications and approaches that fall under the umbrella of this interdisciplinary field.

In today's modern healthcare system, biomedical signal processing is an essential component that encompasses the analysis and interpretation of physiological signals that are produced by the human body. These signals, which include electrocardiograms (ECG), electroencephalograms (EEG), electromyograms (EMG), and several others, give essential insights into the functioning of diverse biological systems. They are necessary instruments for diagnosing, monitoring, and treating a broad variety of health disorders, and they function in this capacity perfectly. Specifically, electrocardiogram (ECG) signals have a significant role in the identification of cardiac abnormalities such as arrhythmias and myocardial infarction. On the other hand, electroencephalogram (EEG) data contribute to the diagnosis of neurological illnesses such as epilepsy and sleep disorders. In addition, respiratory signals are helpful in determining the presence of respiratory illnesses such as chronic obstructive pulmonary disease (COPD) and sleep apnea. In addition to diagnosis and monitoring, the significance of biomedical signal processing in the healthcare industry transcends these two areas. The optimisation of treatment techniques, the facilitation of disease prevention, the enablement of personalised medicine, and the drive of biomedical research and development are all additional contributions that benefit from this. By utilising modern signal processing techniques, healthcare practitioners are able to make judgements based on accurate information, to adjust therapies to the specific requirements of particular patients, and to increase medical knowledge, eventually leading to improved healthcare outcomes for patients all around the world.

In the field of mathematics, fractional calculus is a subfield that extends the conventional concepts of differentiation and integration to orders that are not integers. In contrast to classical calculus, which is concerned with derivatives and integrals of integer orders (for example, first, second, or higher-order derivatives), fractional calculus is concerned with operators that operate on functions that have orders that are not integers. Fractional calculus is useful in signal processing because it can capture complicated behaviours and long-range dependencies that are found in many real-world signals. This capacity is what makes fractional calculus relevant. The conventional methods of signal processing, which are founded on the concept of integer-order calculus, might not be able to accurately characterise signals that possess fractal or non-local features. On the other hand, fractional calculus offers a framework that is more readily adaptable for the analysis and processing of signals of this kind.

When it comes to biological signals, for instance, they frequently display non-stationary and self-similar features that are not capable of being fully captured by traditional signal processing techniques. The use of fractional calculus provides a strong toolkit that may be utilised for the purpose of analysing and obtaining information that is useful from these signals. The use of fractional derivatives and integrals allows researchers to more correctly describe the fractional dynamics of physiological processes. This, in turn, leads to enhanced diagnostic capabilities and a better knowledge of the physiological mechanisms that lie under the surface. There are many different signal processing activities that can benefit from the use of fractional calculus. Some of these jobs include noise reduction, feature extraction, time-frequency analysis, and signal denoising procedures. Researchers and practitioners in the field of signal processing might improve the efficiency of biomedical signal processing systems and gain new insights into complicated biological events by combining fractional calculus approaches into signal processing algorithms. From a broad perspective, the incorporation of fractional calculus into signal processing approaches offers the potential to enhance our capacity to analyse, interpret, and modify signals in a variety of domains, such as biomedical engineering, telecommunications, and geophysics.

Understanding Fractional Calculus

Calculus with non-integer orders of derivatives and integrals is known as fractional calculus. The derivatives and integrals in fractional calculus are expanded to encompass fractional orders, in contrast to classical calculus that only considers integrals and derivatives of integer orders. Physics, engineering, biology, and even finance may all benefit from understanding fractional calculus. Complex systems with memory effects, long-range relationships, and complicated dynamics may be understood and modelled using the fundamental framework of fractional calculus. It provides a more sophisticated way of expressing things that can't be described using just integrals and derivatives of integer order. The fractional derivative, represented as \(\frac{d^\alpha}{dx^\alpha} \), is an essential idea in fractional calculus. Here, \(\alpha \) can be a real or complex number. To characterise systems with fractional or fractal dynamics, the fractional derivative extends the idea of differentiation to non-integer orders. Similarly, the idea of integration may be expanded to non-integer orders, represented by \(\int^\alpha \), by fractional integrals. A broad variety of fields make use of fractional calculus. Anomalous diffusion, viscoelastic materials, and non-local events are modelled using fractional calculus in the physical sciences. Optimisation, control systems, and signal processing are some of the technical domains that make use of it. Anatomy, neuronal dynamics, and physiological transport are all better understood with the use of fractional calculus in the medical and biological sciences. Fractional calculus also has financial applications in the areas of stochastic process modelling and option pricing. Familiarity with the fundamental ideas, characteristics, and uses of fractional calculus is necessary for its comprehension. Learning the ins and outs of integrals and fractional derivatives, along with their mathematical features and connections, is essential. Expertise in fractional calculus typically necessitates hands-on experience in solving issues across several disciplines using its concepts. A robust mathematical toolbox for describing complex systems and phenomena with fractional-order dynamics is provided by fractional calculus. Researchers and practitioners from all walks of life can benefit from studying it because of its multidisciplinary character and broad range of applications.

Fractional Calculus in Image Denoising:

A powerful approach for image denoising is provided by fractional calculus. This method is useful for dealing with derivatives and integrals of orders that are not integers. Because of this, denoising algorithms are able to capture long-memory and self-similar characteristics in images. As a consequence, they are able to reduce noise more effectively while still preserving essential image information. The most significant advantage is that it is adaptive, which enables denoising algorithms to dynamically adjust their behaviour dependent on the characteristics of the noise and the content of the image. This is the most crucial benefit. The formula for denoising, which is derived from fractional calculus, may be expressed in the following manner:

$$D^{\alpha} f(x,y) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \, d\tau$$

The fractional derivative of the image $f(x, y)$ of order α is denoted by the symbol H$\alpha f(x, y)$. The gamma function is denoted by $\Gamma(n - \alpha)$, and n is the smallest integer that is bigger than α.

The mathematical idea known as fractional calculus is an extension of the classical calculus that may be used to orders of differentiation and integration that are not integers. Numerous industries, including signal and image processing, have discovered uses for it in their respective applications. In the process of picture denoising, fractional calculus may be utilised to develop filters that efficiently improve noise reduction while maintaining the integrity of essential image characteristics. The Fractional Order Total Variation (FOTV) denoising method is one of the denoising approaches that is based on fractional calculus and is frequently utilised. When compared to the conventional Total Variation (TV) denoising strategy, the FOTV method incorporates a fractional order factor into the equation. When it comes to denoising, the Total variance (TV) approach seeks to minimise the total variance of a picture while simultaneously lowering noise. In other words, it may be stated as follows:

$$Minimize: TV(u) = \int\int |\nabla u| \, dxdy$$

where the denoised picture is denoted by u, the gradient of the image is denoted by ∇m, and the magnitude of the gradient is signified by $|\nabla m|$.

The incorporation of fractional calculus is achieved by introducing a fractional order α ($0 < \alpha < 1$) into the TV denoising technique. This results in the creation of the Fractional Order Total Variation (FOTV) denoising method. One possible representation of the FOTV denoising approach is as described below:

$$Minimize: FOTV(u) = \int\int |\nabla u| \, \alpha dxd$$

The fractional order parameter is denoted by the symbol α. The classic TV denoising method may be reduced to the FOTV denoising method when the value of α is equal to 1.

Example:

In this example, we will use the FOTV denoising method to remove noise from a grayscale image that contains noise. Let's say we have the following image, which is really noisy:

Noisy Image:

To denoise this image using FOTV with $\alpha = 0.5$, we perform the following steps:

Step 1: Calculate the gradient magnitude of the image:

Step 2: Raise the gradient magnitude to the power of α:

Step 3: Minimize the FOTV functional by finding the denoised image u that minimizes

$$\int\int |\nabla u|^{\alpha} \, dxdy$$

through the maintenance of essential aspects of the image. Denoising technique known as the Fractional Order Total Variation (FOTV) approach:

$$FOTV(u) = \int\int |\nabla u|^{\alpha} \, dxdy$$

where:

Fractional Order Total Variation is used to express the denoising functional, which is denoted by the symbol m(x). This is the image that has been denoised. The gradient of the picture, which represents the fluctuations in spatial intensity, is denoted by the symbol T. "Edge strength" is represented by the magnitude of the gradient of the picture, which is denoted by the symbol "Edge." A fractional order parameter, denoted by the symbol α, is responsible for determining the level of smoothness and edge preservation associated with the denoised picture. It ought to be a number that falls anywhere between 0 and 1.

The purpose of the FOTV denoising approach is to provide effective noise reduction while keeping significant picture characteristics. This is accomplished by minimising the value of the FOTV functional. When performing the minimization procedure, it is necessary to locate the denoised picture u that fulfils the equation shown above and generates the optimal balance between the elimination of noise and the retention of features. In order to discover the denoised image that is the best possible, the actual implementation of this approach entails the use of numerical optimisation techniques. It is important to note that the actual implementation of the FOTV denoising technique entails the use of numerical methods, such as iterative optimisation, in order to locate the denoised picture u that solves the minimization issue described above.

When compared to the traditional integer-order methods, the fractional calculus provides a strong toolkit for picture denoising and other signal processing applications. This toolkit enables denoising approaches that are more flexible and resilient. However, it is important to note that the implementation of these approaches may be more difficult and computationally intensive owing to the presence of fractional derivatives. This factor should be taken into consideration. Additionally, fractional calculus makes it possible for denoising algorithms to manage non-local dependencies in pictures, which allows for the effective preservation of texture and border lines. It is still difficult to find a solution to the problem of dealing with the computational complexity that is connected with fractional integrals and derivatives. The goal of future research is to build efficient algorithms that will guarantee real-time denoising performance and harness the full potential of fractional calculus in picture denoising.

Fractional Calculus in Filtering for Signal Processing

In recent years, fractional calculus has emerged as a potentially useful method in the field of signal processing, particularly in the area of filtering. When dealing with non-stationary signals or signals with variable properties, traditional filtering approaches that are based on integer-order calculus may be subject to constraints. The framework of dynamic and adaptive filtering that is provided by fractional calculus makes it possible to conduct signal analysis and processing that is more efficient. One of the most significant benefits of employing fractional calculus in filtering is that it has the capability to modify its behaviour in response to changes in the characteristics of the signal. This is because traditional filters often have features that are fixed, which makes them less ideal for managing signals that have properties that change over time. Alternatively, filters that are based on fractional calculus have the ability to modify their response in accordance with the changing characteristics of the input signal, which ultimately results in filtering results that are more accurate and reliable.

The fractional calculus-based filtering formula can be represented as:

$$D^{\alpha}x(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{dt}^{\mathrm{n}}} \int_{a}^{t} \frac{\mathrm{x}(\tau)}{(\mathrm{t}-\tau)^{\alpha-\mathrm{n}+1}}\, \mathrm{d}\tau$$

The fractional derivative of the signal $x(t)$ of order α is denoted by $\boldsymbol{\sigma}\alpha x(t)$. The gamma function is represented by $\Gamma(n-\alpha)$, and n is the smallest integer that is bigger than α. Filtering algorithms are able to capture long-memory relationships and self-similar patterns in signals by utilising fractional derivatives. This enables them to be well-suited for the processing of complicated and non-stationary signals.

Utilising fractional calculus allows for the creation of fractional order filters, which have been demonstrated to be useful in a variety of applications across the board. An example of a fractional order filter that is often utilised is the Fractional Order Low-Pass Filter, also known as the FOLPF. Within the context of the continuous-time domain, the FOLPF is expressed as follows: A fractional order low-pass filter (FOLPF) in the time domain can be represented by the following equation:

$$y(t) = \int x(\tau).(t - \tau)^{\alpha-1}\, d\tau$$

where:

$y(t)$ is the output signal (filtered signal) at time t.

$x(\tau)$ is the input signal at time τ.

α is the fractional order parameter, which determines the characteristics of the filter response. It should be a positive value ($0 < \alpha < 1$ for a low-pass filter).

The above equation represents a fractional integral of the input signal $x(\tau)$ with respect to the variable τ, where the integration order is determined by α.

Example:

Let's apply the Fractional Order Low-Pass Filter (FOLPF) to a simple example signal in the time domain.

Consider the following input signal:

$x(t)$ = [0, 1, 2, 3, 4, 3, 2, 1, 0]

Assume the fractional order (α) of the filter to be 0.7.

Step 1: Apply the FOLPF equation to filter the input signal.

For each time step t, we perform the integration using the given equation:

$$y(t) = \int x(\tau) * (t - \tau)^{\alpha-1} \, d\tau$$

Step 2: Calculate the filtered output signal $y(t)$ for each time step.

$y(1) = 0 * (1 - 0)\ 0.7{-}1 = 0$

$y(2) = (0 * (2 - 1)\ 0.7{-}1 + 1 * (2 - 2)\ 0.7{-}1\ ) = 1$

$y(3) = (0 * (3 - 1)\ 0.7{-}1 + 1 * (3 - 2)\ 0.7{-}1 + 2 * (3 - 3)\ 0.7{-}1\ ) \approx 1.951$

$$y(4) = (0 * (4 - 1)^{0.7-1} + 1 * (4 - 2)^{0.7-1} + 2 * (4 - 3)^{0.7-1} + 3 * (4 - 4)^{0.7-1}) \approx 3.400$$

... and so on for other time steps.

Step 3: The resulting $y(t)$ values represent the filtered output signal.

Please take into consideration that the example presented above is only a simplified instance. In real-world applications, fractional order filtering is frequently performed through the use of numerical approaches or discrete approximations in order to get results that are both efficient and accurate. As a result of its capacity to capture noninteger order dynamics, fractional order filters may also be utilised in signal processing applications for the purpose of performing tasks such as noise reduction, signal smoothing, and feature extraction. However, there are also certain difficulties associated with the utilisation of fractional calculus in the filtering process. The computational complexity that is associated with fractional derivatives is a significant problem that may necessitate the use of specialised algorithms and optimisation techniques in order to guarantee fast processing, particularly for systems that operate in real time. As a conclusion, fractional calculus offers a viable and adaptable solution to the filtering process in signal processing. Filters that are based on fractional calculus offer increased performance in the handling of non-stationary signals and contribute to the improvement of signal processing techniques. This is because they enable dynamic modifications to changing signal properties. It is anticipated that the capabilities and uses of fractional calculus in filtering for signal processing will be further improved as a result of ongoing research and development in this field.

Fractional Calculus in Time Series Analysis for Signal Processing:

The use of fractional calculus in the field of time series analysis for signal processing entails the utilisation of fractional derivatives and integrals for the purpose of analysing and processing time series data. The following is a sampling of important equations and formulae, along with an illustration:

Fractional Order Differentiation:

Using the Caputo definition, the fractional derivative of a function $f(t)$ with respect to t of order α is represented by the symbol $D\alpha f(t)$, and it may be stated in the following manner:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{n-\alpha-1}} \, d\tau$$

where: n is the smallest integer greater than α.

PROBLEM

In this study, a systematic approach will be taken in order to overcome the primary challenge that arises when employing fractional calculus in the field of biomedical signal processing. The current research on the subject will be analysed, beginning with a comprehensive assessment of the relevant literature, in order to identify areas that require more inquiry and gaps in the research. After that, a crystal clear problem statement will be established, which will outline the particular research issue and aims within the framework of biomedical signal processing. Following this, a framework will be established, which will outline the procedures needed in applying concepts from fractional calculus to the processing and analysis of biological data. This technique will include the gathering of data, preparation of the data, analysis of the data, and interpretation of the data, as well as the incorporation of corresponding algorithms and models as required. This will be followed by the collection and analysis of relevant biological signal datasets using the technique that has been described, with the primary focus being on the extraction of significant insights and the evaluation of the success of the strategy by statistical or computational studies. Visualisations and other ways that include illustrative methods will be utilised in order to communicate the outcomes of the study, which will include both the experimental findings and the computational results. This will be followed by an interpretation and discussion of these results in connection to the primary issue, with an emphasis on the implications that they have for biological signal processing and a discussion of any difficulties or constraints that were encountered throughout the course of the research process. The last section of the study will conclude by providing a summary of the most important findings, highlighting the relevance of those discoveries, and recommending prospective routes for further research or the refining of fractional calculus techniques in the field of biomedical signal processing. The appropriate citation of pertinent literature and sources shall be maintained throughout the entirety of the work in order to ensure that the contributions of earlier study in the subject are acknowledged.

METHOD

ic signal created by combining the pulse signal (R-waves of ECG) with the electromyographic signal, as described in greater detail in.Two electrodes were positioned across the left side of the subject's chest, and a pulse rate signal was captured from within those electrodes. In accordance with the SENIAM protocol, electrodes for electromyography were positioned across the proximal bulk of the Trapezius muscle belly location. Both signals were cropped to a period of twenty seconds after the recording was completed, and then they were superimposed in order to capture both the pulse rate signal that was contaminated with EMG and the EMG signal that was contaminated with the pulse rate signal.

Preprocessing was performed on the semi-synthetic signal that was obtained by employing a Butterworth band-stop third order filter with cuts at frequencies of 47 Hz and 53 Hz. This was done in order to decrease the power noise. After applying the first median filter with a window width of 90 milliseconds, we proceeded to apply the FC filter in order to eliminate the EMG artefact. We began by using FC, and then we moved on to using a median filter with a window width of 400 milliseconds in order to extract the pulse rate signal from the EMG. Three, six, nine, and one (the usual first order derivative) are the orders of fractional FC that were utilised in this study. Calculating the signal-to-noise ratio (SNR) for the signal after applying simply the median filter and FC coupled with the median filter was the method that was utilised in order to evaluate the suggested strategy.

RESULTS

Table 1 displays the signal-to-noise ratio (SNR) values for signals that have been subjected to further applications of fractional derivatives of varying orders on the median filter, as well as for signals that have only been subjected to the median filter. Extraction of EMG noise from pulse rate signal and extraction of pulse rate noise from EMG signal are both included in these results, which pertain to both applications. For the median filter, the window lengths were 90 milliseconds and 400 milliseconds, respectively.

Table 1. The signal-to-noise ratio (SNR) values for the signal after the application of the median filter were compared to those for the signal after the application of several orders of fractional derivative in addition to the median filter.

Fractional Order	Pulse rate signal EMG noise	EMG signal Pulse rate noise
0 (median filter only)	3.4 dB	25.4 dB
0.3	18.4 dB	40.3 dB
0.6	31.5 dB	53.5 dB
0.9	37.8 dB	58.3 dB
1	33.5 dB	51.5 dB

The signal-to-noise ratio (SNR) of the signal was 37.8 dB after the effects of the EMG artefact were eliminated, but it was only 3.4 dB after the median filter was applied. When the pulse rate artefact was removed from the signal, the signal-to-noise ratio (SNR) value was 58.3 dB, and when simply the median filter was used, the SNR value was 25.4 dB.

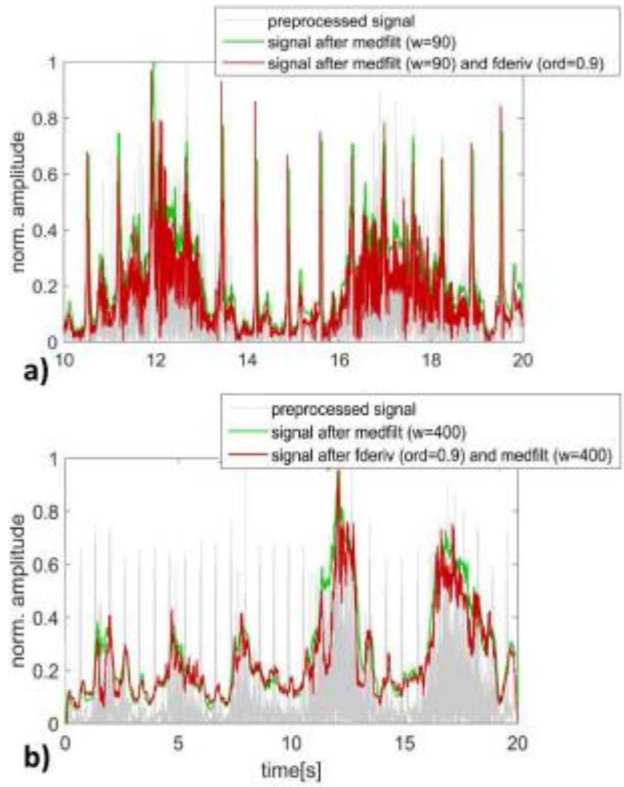

Figure 2. In order to maintain a) the heart rate and b) the EMG signal envelope, the preprocessed data were compared with those that had just been applied a median filter or those that had been applied in conjunction with FC. Ordinized is what Norm. stands for.

For example, the preprocessed signal, the signal after the application of the median filter, and the signal after the application of FC with order 0.9 in addition to the median filter are all displayed in Figure 2.

Fundamentals of Partial Derivative Variables

During this section, we will determine whether or not a number of essential properties, including as composition, linearity, and Leibniz's rule, continue to be valid for various integrals.

Consistency

The linearity of the function may be achieved by completely specifying the integrals and fractional derivatives. Through the use of the formula for the Grunwald-Letnikov fractional derivative, we may be able to get.

$$
\begin{aligned}
{}_aD_t^p\Big(\lambda f(t)+\mu g(t)\Big) &= \lim_{\substack{h\to 0\\ mh=t-a}} h^{-p}\sum_{r=0}^{m}(-1)^r\binom{p}{r}\Big(\lambda f(t-rh)+\mu g(t-rh)\Big)\\
&= \lambda \lim_{\substack{h\to 0\\ mh=t-a}} h^{-p}\sum_{r=0}^{m}(-1)^r\binom{p}{r} f(t-rh)\\
&\quad + \mu \lim_{\substack{h\to 0\\ mh=t-a}} h^{-p}\sum_{r=0}^{m}(-1)^r\binom{p}{r} g(t-rh)\\
&= \lambda\, {}_aD_t^p f(t) + \mu\, {}_aD_t^p g(t).
\end{aligned}
$$

The functions f(t) and g(t) are specifically defined by the operator that is provided in this proof. Additionally, the real constants λ and μ are derived from the set R. When it comes to the fractional integral, it is also feasible to demonstrate the same proposition.

In addition to this, we will provide evidence about the linearity of the Riemann-Liouville divergence integral.

$$\begin{aligned} {}_aD_t^{-p}\Big(\lambda f(t)+\mu g(t)\Big) &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\Big(\lambda f(\tau)+\mu g(\tau)\Big)\,\mathrm{d}\tau \\ &= \lambda\frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1} f(\tau)\,\mathrm{d}\tau \\ &\quad + \mu\frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1} g(\tau)\,\mathrm{d}\tau \\ &= \lambda\ {}_aD_t^{-p}f(t)+\mu\ {}_aD_t^{-p}g(t). \end{aligned}$$

There is the possibility of providing the same proof for the Rieman-Liouville derivative here. By way of illustration, let's use the definition of the Riemann-Liouville integral and the linearity that was only recently shown.

Minimal Policy

We are able to deduce the following: assuming that f(t) is continuous for any values of t that are larger than or equal to a

$$\lim_{p\to 0}\ {}_aD_t^{-p}f(t) = f(t).$$

Discovering the proof is not a difficult task. Because of this, we will define

$${}_aD_t^{0}f(t) = f(t).$$

Leibniz's Rule and the Product Rule

Since f and g are functions, we are aware that the product rule is the formula that calculates the derivative of their product. This is because the product rule is a formula.

$$(f\cdot g)' = f'\cdot g + f\cdot g'.$$

It is possible to generalize this to

$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(k)} g^{(n-k)},$$

the Leibniz rule is another name for this particular model. In the last sentence, the functions f and g may each be differentiated n times. Both of these functions are functions. Taking into consideration that both f(τ) and g(τ) are continuous functions on the interval [a, t], we are able to establish that the formula that follows provides the Leibniz rule for fractional derivatives.

$$ {}_aD_t^p\Big(f(t)g(t)\Big) = \sum_{k=0}^{m} \binom{p}{k} f^{(k)}(t)\ {}_aD_t^{p-k} g(t), $$

For m belonging to the set N, the statement "m ≤ p < m+1" holds true, and the binomial coefficient is once again the same. Even though it is extremely long, I will not provide the proof here; nonetheless, you are able to locate it. If we already know the fractional derivative of one function, g(t), and we want to know the fractional derivative of the function that is the result of adding another function, f(t), then Leibniz's method is a valuable tool for obtaining the fractional derivative of a function that is a product of two functions. This is because the fractional derivative of a function is the product of two functions.

Differential equations that are fractional and linear

The differential equation may be extended to include fractional differential equations under certain circumstances. When it comes to finding solutions to these problems, the Laplace transform is only one of many viable answers. Let's go over the principles of the Laplace transform so that you can follow along with the rest of the chapter. We will discuss this technique later on, but before we do so, let's go over the fundamentals of the Laplace transform.

The Riemann-Liouville Differ integral's Laplace transform

Our first order of business will be to study the Laplace transform of the Riemann-Liouville fractional integral.

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau)\, \mathrm{d}\tau. $$

The final statement may be rewritten as follows if we define the function g(t) = t p−1 and apply the convolution definition.

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} t^{p-1} * f(t) = \frac{1}{\Gamma(p)} g(t) * f(t) = \frac{1}{\Gamma(p)} (g * f)(t). $$

A look at the Laplace transform of g(t) and the application of the definition of the Laplace transform

$$ G(s) = L\{g(t); s\} = L\{t^{p-1}; s\} = \int_0^\infty t^{p-1} e^{-st}\, \mathrm{d}t. $$

Making the substitution of st for r results in the equation dt = 1 s dr, which enables us to rewrite the final expression as

$$ G(s) = \frac{1}{s^p} \int_0^\infty r^{p-1} e^{-r}\, \mathrm{d}r = s^{-p} \int_0^\infty r^{p-1} e^{-r}\, \mathrm{d}r = \Gamma(p) s^{-p}, $$

At this point, the Laplace transform of the Riemann-Liouville fractional integral may be specified appropriately.

$$ L\{{}_0D_t^{-p} f(t); s\} = L\Big\{ \frac{1}{\Gamma(p)} (g * f)(t); s \Big\}. $$

$$ L\{{}_0D_t^{-p} f(t); s\} = \frac{1}{\Gamma(p)} G(s) F(s). $$

These uncommon circumstances may prove to be helpful in the resolution of a few fundamental fractional differential equations, as we will see in the examples that are presented at the end of this chapter.

CONCLUSION

In conclusion, the research presented in this article has investigated the revolutionary effect that fractional calculus has had on the area of signal processing. A powerful and novel technique that offers a more flexible and accurate representation of complex signals is fractional calculus, which deals with derivatives and integrals of non-integer orders. This enables fractional calculus to be used in a variety of contexts. Particularly noteworthy uses of fractional calculus in signal processing include picture denoising, filtering, and time series analysis. These applications have been particularly remarkable. picture denoising approaches have achieved impressive achievements by virtue of their utilisation of fractional derivatives and integrals. These techniques have been able to maintain vital picture information while simultaneously efficiently lowering noise levels. The use of fractional calculus to filter signals has resulted in the introduction of a dynamic and adaptable technique, which contributes to the improvement of signal analysis even in contexts that are constantly changing. Furthermore, fractional calculus has tremendously enhanced time series analysis by enabling more accurate modelling of time-dependent data and boosting prediction skills. This has greatly contributed to the improvement of time series analysis. Through the incorporation of essential formulae and notations associated with fractional calculus, a foundation has been established for comprehending the theoretical elements of this mathematical concept as well as its practical use in signal processing applications. Fractional calculus is destined to play an increasingly vital part in signal processing approaches as technology continues to improve at a rapid pace. The fact that it is able to capture long-memory and self-similar features in real-world signals distinguishes it from the conventional integer-order calculus and opens up new paths for the investigation and interpretation of complex events. Despite the fact that fractional calculus offers a great deal of potential, there are still obstacles to overcome in terms of the difficulty of computing and the creation of efficient algorithms. In spite of this, it is anticipated that continued research and innovation in this sector will be able to solve these problems and unlock the full potential of fractional calculus in signal processing.

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